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# Realizations of perturbations of an observable pair with prescribed indices ☆

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ABSTRACT

It is well known that, when a full rank observable pair $(C, A)$ is slightly perturbed, the new observability indices $k'$ are majorized by the initial ones $k$, $k \succ k'$. Conversely, any indices $k'$ majorized by $k$ can be obtained by perturbing $(C, A)$. The aim of this paper is the explicit construction of perturbations of $(C, A)$ which have the desired indices $k'$ by means of a sequence of uniparametrical versal perturbations. Even more, using versal deformations we refine this construction in such a way that the perturbation has the maximum possible number of zeros and no parameters in the square part.


## 1. Introduction

It is well known [4,2] that, when a full rank observable pair $(C, A) \in \mathbb{C}^{m\times n} \times \mathbb{C}^{n\times n}$, with observability indices $k = (k_1, k_2, \ldots, k_m)$ in non-increasing order, is slightly perturbed, the new observability

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indices $k' = (k'_1, k'_2, \ldots, k'_m)$ in non-increasing order are *majorized* [5] by the initial ones $k$, that is to say:

$$k_1 \geqslant k'_1, k_1 + k_2 \geqslant k'_1 + k'_2, \ldots, \sum_{j=1}^{m} k_j = \sum_{j=1}^{m} k'_j.$$

Conversely, any indices $k'$ majorized by $k$ can be obtained by perturbing $(C, A)$.

The aim of this paper is the explicit construction of "minimal"perturbations of $(C, A)$ having the desired indices $k'$, where "minimal"means that one preserves as many entries of $(C, A)$ as possible. We can guarantee this "minimality" because the only allowed perturbations of $(C, A)$ are those in the miniversal deformation in Theorem 2.4 [1]: The number of parameters is minimal according to Arnold's theory and each one perturbs a unique entry of $(C, A)$.

The construction is attempted in two steps. First (Theorem 3.7), we obtain a realization of $k'$, that is to say, an explicit perturbation of $(C, A)$ having the desired indices $k'$. Next (Theorem 4.11), we move the parameters appearing there in order to place them in the entries in the miniversal deformation in Theorem 2.4 [1]. Again Arnold's theory ensures that this replacement is possible, but the explicit construction is not trivial.

In Section 2 we establish the notation, we recall what we understand by BK-canonical form of a pair (Definition 2.2), what a miniversal deformation is and Theorem 2.4, which gives the miniversal deformation obtained in [1].

Finally, we define what we mean by elementary versal perturbation, realization and versal realization of any tuple of indices.

Section 3 is devoted to obtaining realizations of tuples of indices majorized by the given one, by means of a (good) sequence of the so-called elementary ones.

First, we study the elementary versal perturbations. We see that the new observability indices $k'$ differ from the initial ones in only two of them: $k' = (\ldots, k_i - p, \ldots, k_j + p, \ldots)$. We say that $k'$ is an elementary change of $k$.

For any elementary versal perturbation, one computes explicitly the change of basis that reduces this versal perturbation to its BK-canonical form.

For a general tuple of indices $k^f$ majorized by $k$, we consider a sequence of $m$-tuples of indices

$$k^{(0)} = k, k^{(1)}, k^{(2)}, \ldots, k^{(l)} = k^f$$

of elementary changes, that is to say: each $k^{(j)}$ is majorized by the previous one $k^{(j-1)}$ and both differ only in two indices. Thus, each $k^{(j)}$ can be realized as an elementary versal perturbation of the previous one. In order to simplify the computations, we restrict ourselves to sequences of this kind having minimal length, which we call good sequences.

Thus, given a full rank observable pair $(C, A)$, with observability indices $k = (k_1, k_2, \ldots, k_m)$ and a tuple of indices $k^f$ majorized by $k$, we construct a realization of $k^f$ by means of elementary versal perturbations in a good sequence (Theorem 3.7).

We notice through an example that this realization is not good in the sense that the parameters are located in both matrices.

In order to correct it, the remainder Section 4 is devoted to changing the intermediate perturbations in order to obtain a final realization with all the parameters in the entries of the initial miniversal deformation (Theorem 4.11 and Corollary 4.12).

## 2. Preliminaries

Let $\mathcal{M} = \{(C, A) : A \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{m\times n}\}$ be the differentiable manifold of pairs of matrices and let $\mathcal{M}^*$ be the open dense subset of $\mathcal{M}$ formed by the observable pairs with rank $C = m$, that is to say, the *full rank observable pairs*.

The usual block similarity (or BK-equivalence) is induced by the following group action:

$$\mathcal{G} = \left\{ g = \begin{pmatrix} G_1 & G_2 \\ 0 & G_3 \end{pmatrix} : G_1 \in Gl_n, G_3 \in Gl_m, G_2 \in \mathbb{C}^{n\times m} \right\},$$

$$g * (C, A) = (G_3 C G_1^{-1}, G_1 A G_1^{-1} + G_2 C G_1^{-1}).$$

So, the BK-equivalence class of a pair $(C, A)$ is the orbit

$$O_{BK}(C, A) = \{g * (C, A) : g \in \mathcal{G}\}.$$

The pair $(C, A)$, written in vertical form $\begin{pmatrix} A \\ C \end{pmatrix}$, can be interpreted as the matrix of a linear map from $\mathbb{C}^n$ to $\mathbb{C}^{n+m}$ considering in the second space an adapted basis to the first one. Then, the action of the group is a change of basis preserving $\mathbb{C}^n$.

It is well known that the observability indices $k = (k_1, \ldots, k_m)$ form a complete family of invariants for this equivalence relation and that a canonical form can be associated with each pair.

We fix some notation for the sequel.

**Notation 2.1**

1. $E_q = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1-valued entry is in the q-position and the size corresponds to the context, $e_q = E_q^t$.
2. $N_p = (e_2, \ldots, e_p, 0) = \begin{pmatrix} 0 & 0 \\ I_{p-1} & 0 \end{pmatrix}$ is the lower nilpotent $p$-block.

**Definition 2.2** [3]. Given a full rank observable pair of matrices $(C, A) \in \mathcal{M}^*$ and $k_1, k_2, \ldots, k_m$ its observability indices, the pair

$$(C_c, A_c) = (diag(E_{k_1}, E_{k_2}, \ldots, E_{k_m}), diag(N_{k_1}, N_{k_2}, \ldots, N_{k_m}))$$

is called its *BK-canonical form*(determined except for a permutation of the indices). An adapted basis to the subspace in which the linear map has this matrix, written in vertical form, is known as a *BK-basis*.

**Notation 2.3.** Following this pattern we will consider the following block-partitions:

$$M \in \mathbb{C}^{n\times n}, \quad M = (M_{i,j})_{1\leqslant i,j\leqslant m}, \quad M_{i,j} \in \mathbb{C}^{k_i\times k_j}.$$
$$M \in \mathbb{C}^{n\times m}, \quad M = (M_{i,j})_{1\leqslant i,j\leqslant m}, \quad M_{i,j} \in \mathbb{C}^{k_i\times 1}.$$
$$M \in \mathbb{C}^{m\times n}, \quad M = (M_{i,j})_{1\leqslant i,j\leqslant m}, \quad M_{i,j} \in \mathbb{C}^{1\times k_j}.$$

When 0 appears in a block matrix, it will be a null block of the suitable size (it could be empty).

We will denote by $M^j$ the $j$th row and by $(M)_i$ the $i$th column.

Given an m-tuple of indices $k = (k_1, k_2, \ldots, k_m)$ we write $k_i \in k$ and its length as $l(k) = m$. If $k'$ is another m-tuple of indices majorized by $k$ we write $k \succ k'$.

We denote by $l_j \doteq \sum_{i\leqslant j} k_i$.

According to Arnold's techniques, any equivalence class near a point is represented in the so-called versal deformations, which can be characterized as differentiable manifolds transverse to the corresponding orbit. By miniversal deformation we understand versal with minimum dimension. This geometric structure for the BK-equivalence has been studied in [1]. In particular, for a full rank observable pair, we have the following theorem.

**Theorem 2.4** [1]. *Given a full rank observable pair $(C, A) \in \mathcal{M}^*$ with observability indices $k_1, k_2, \ldots, k_m$, then*

$$\dim O_{BK}(C, A) = n^2 + nm - \sum_{i,j=1}^{m} max\{0, k_j - k_i - 1\}.$$

*If $(C_c, A_c)$ is its BK-canonical form, a BK-miniversal deformation in $\mathcal{M}^*$ of it is the linear manifold $\{(C_c + V_\alpha, A_c)\}_\alpha$, where $(V_\alpha)_{i,j} = \sum_{1 \leqslant l \leqslant k_{i,j}} \alpha_{i,j,l} E_{k_j - l}$, with $k_{i,j} = max\{0, k_j - k_i - 1\}$.*

**Example 2.5.** If $k_1 = 6, k_2 = 4, k_3 = 2, k_4 = 1$, then

$$V_\alpha = \left(\begin{array}{cccccc|cccc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha_{2,1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \alpha_{3,1,3} & \alpha_{3,1,2} & \alpha_{3,1,1} & 0 & 0 & 0 & \alpha_{3,2,1} & 0 & 0 & 0 & 0 \\ \hline 0 & \alpha_{4,1,4} & \alpha_{4,1,3} & \alpha_{4,1,2} & \alpha_{4,1,1} & 0 & 0 & \alpha_{4,2,2} & \alpha_{4,2,1} & 0 & 0 & 0 & 0 \end{array}\right).$$

From now on we will enumerate the indices of the parameters as in this example.

**Definition 2.6.** Let $(C, A)$ be a full rank observable pair, let $k$ be its observability indices and let $\{(C + V_\alpha, A)\}_\alpha$ be the miniversal deformation in Theorem 2.4 [1].

1. The pairs $(C + V_\alpha, A)$ obtained when $\alpha \neq 0$ will be called *miniversal perturbations* of $(C, A)$. In particular, we say that the pair $(C + V_\alpha, A)$ is *elementary* if only one of the parameters in $\alpha$ is non-zero.
2. Given a tuple of indices $k^f$ majorized by another $k$, any miniversal perturbation of $(C, A)$ having $k^f$ as observability indices will be called a *miniversal realization* of $k^f$ (from $k$).

**Remark 2.7.** Notice that the miniversal deformation in Theorem 2.4 is "minimal"in the sense that the number of parameters is just the codimension of $O_{BK}(C, A)$ and that each one appears only one time in $V_\alpha$, so that, the number of preserved entries of $(C, A)$ is maximal. In this sense, the above miniversal realizations are "minimal". In fact, the matrix $A$ is fully preserved.

## 3. Iterative elementary realizations

To simplify the notation, we will change the order of the observability indices, in such a way that the elementary perturbation concerns the first two indices.

**Proposition 3.1.** *Let $(C, A)$ be a pair in BK-canonical form and let $k = (k_1, k_2, \ldots, k_m)$ be the tuple of its observability indices with $k_1 > k_2, k_3 \geqslant \cdots \geqslant k_m$. Let $(C_1, A_1)$ be a pair in BK-canonical form and let $k' = (k_1 - p, k_2 + p, k_3, \ldots, k_m)$ be the tuple of its observability indices.*

*If $(C + V_\alpha, A)$ is the elementary perturbation obtained from the miniversal one in Theorem 2.4 taking as unique parameter $\alpha \doteq \alpha_{2,1,p}$, $1 \leqslant p < k_1 - k_2$, then:*

1. *The pair $(C + V_\alpha, A)$ has the tuple of observability indices $k'$.*
2. *The columns of $g = \begin{pmatrix} G_1 & G_2 \\ 0 & G_3 \end{pmatrix}$, where:*

$$G_1 = diag\left(\left(\begin{array}{c|cc} I_{k'_1} & \begin{matrix}0\\ I_{k_2}\end{matrix} & \begin{matrix}0\\ 0\end{matrix} \\ \hline 0 & \begin{matrix}0\\ -\alpha I_{k_2}\end{matrix} & \begin{matrix}I_p\\ 0\end{matrix}\end{array}\right), I_{n-k_1-k_2}\right),$$

$$G_2 = e_{k'_1+1}E_1 \in \mathbb{C}^{n\times m},$$

$$G_3 = diag\left(\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, I_{m-2}\right),$$

*form a BK-basis of the pair* $(C + V_\alpha, A)$.

3. $(C_1, A_1) = g^{-1} * (C + V_\alpha, A)$.

**Proof.** It is sufficient to prove that $g\begin{pmatrix} A_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} A \\ C + V_\alpha \end{pmatrix} G_1$.

Since the only changes between $(C + V_\alpha, A)$ and $(C_1, A_1)$, considered in vertical form, are in the first $l_2$ columns, we can restrict ourselves to calculate them. Just multiplying, the first $l_2$ columns of the resulting matrices in both sides of the equality to prove are:

$$\left(\begin{array}{c|cc} N_{k'_1} & \begin{matrix}0\\ N_{k_2}\end{matrix} & \begin{matrix}0\\ 0\end{matrix} \\ \hline e_1E_{k'_1} & \begin{matrix}e_1E_{k_2}\\ -\alpha N_{k_2}\end{matrix} & \begin{matrix}N_p\\ 0\end{matrix} \\ \hline 0_{(n-l_2)\times k'_1} & 0 & 0 \\ \hline \begin{matrix}0\\ \alpha E_{k'_1}\\ 0_{(m-2)\times k'_1}\end{matrix} & \begin{matrix}0\\ 0\\ 0\end{matrix} & \begin{matrix}E_p\\ 0\\ 0\end{matrix}\end{array}\right). \qquad \square$$

**Lemma 3.2.** *With the above notations we have that*

$$G_1^{-1} = diag\left(\left(\begin{array}{c|cc} & 0 & 0 \\ I_{k_1'} & & \\ & 0 & \alpha^{-1}I_{k_2} \\ \hline & 0 & -\alpha^{-1}I_{k_2} \\ 0 & & \\ & I_p & 0 \end{array}\right), I_{n-k_1-k_2}\right).$$

**Proof.** Just multiplying. □

**Remark 3.3.** It can also be expressed by

$$G_1^{-1} = (e_1, e_2, \ldots, e_{k_1'}, e_{l_2-p+1}, \ldots, e_{l_2}, \alpha^{-1}(e_{k_1'-k_2+1} - e_{k_1'+1}), \ldots,$$
$$\alpha^{-1}(e_{k_1'} - e_{l_2-p}), e_{l_2+1}, \ldots, e_n).$$

If $k = (k_1, k_2, \ldots, k_m)$ and $k^f = (k_1^f, k_2^f, \ldots, k_m^f)$ are two tuples of indices with $k \succ k^f$, we want to obtain sequences of elementary changes from $k$ to $k^f$ as short as possible. From now on we consider that $k_i \neq k_j^f$, for all $i, j$ since the coincident indices do not imply any change.

**Proposition 3.4.** *Let $k = (k_1, k_2, \ldots, k_m)$ and $k^f = (k_1^f, k_2^f, \ldots, k_m^f)$ be two tuples of indices with $k \succ k^f$ and for all $i, j$, $k_i \neq k_j^f$. Then, there is a sequence of elementary changes between $k$ and $k^f$ whit less of $m$ terms.*

**Proof.** We prove the proposition by induction. There is nothing to prove for $m = 2$. From $k_1 > k_1^f$, there is $k_i < k_i^f$ for any $1 < i \leqslant m$. Then, if $k_i^f - k_i \leqslant k_1 - k_1^f$ or $k_i^f - k_i > k_1 - k_1^f$, taking $k_1' = k_1 - (k_i^f - k_i)$, $k_i' = k_i^f$ and $k_j' = k_j$ for the rest or $k_1' = k_1^f$, $k_i' = k_i + (k_1 - k_1^f)$ and $k_j' = k_j$ for the rest, respectively, we have a tuple of indices $k'$ that is an elementary change of $k$ and that has one coincident index and $m - 1$ different indices with the ones of $k^f$. □

**Remark 3.5.** Notice that if $k$ and $k^f$ are two tuples of indices as in Proposition 3.4 with $k = k_{(1)} \cup k_{(2)}$, $k^f = k_{(1)}^f \cup k_{(2)}^f$, $k_{(i)} \succ k_{(i)}^f$ and $l(k_{(i)}) = l(k_{(i)}^f)$, then, there is a sequence of elementary changes between $k$ and $k^f$ with less than $m - 1$ terms.

**Definition 3.6.** Let $k, k^f$ be two tuples of indices such that $k \succ k^f$. We define as *good sequences* the sequences of elementary changes with the properties:

1. In every change the major index (of the tuple obtained after the precedent change) different from all the indices of $k^f$ decreases.
2. If an index decreases in a change it will not increase later.
3. If an index increases in a change it will not decrease later.
4. Two indices can only be related in one change.
5. In each change the final value is obtained for one of the indices.

Given a good sequence, we denote by $k_{(c)}$ the tuple formed by the indices $k_i \in k$ that decrease in a change and by $k_{(r)}$ the one formed by the indices $k_i \in k$ that increase in any change. It is obvious that $k_{(c)} \cap k_{(r)} = \emptyset$ and $k_{(c)} \cup k_{(r)} = k$.

We construct a realization of $k^f$ by means of elementary miniversal perturbations in a good sequence.

**Theorem 3.7.** *Let $(C, A)$ be a full rank observable pair and let $k$ be its observability indices. For a general tuple of indices $k^f$ majorized by $k$, we consider a good sequence of $m$-tuples of indices $k^{(0)} = k, k^{(1)}, k^{(2)}, \ldots, k^{(l)} = k^f$. Let $(C, A) \equiv (C_0, A_0), (C_1, A_1), \ldots, (C_l, A_l)$ be the respective BK-canonical forms. We have proved in Proposition 3.1 that each $(C_i, A_i)$ can be realized as an elementary miniversal perturbation of the previous one. Hence, we have single parameters $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$ and changes of basis $g_0, g_1, \ldots, g_{l-1}$ such that: $(C_1, A_1) = g_0^{-1} * (C_0 + V_{0,\alpha_0}, A_0)$, $(C_2, A_2) = g_1^{-1} * (C_1 + V_{1,\alpha_1}, A_1), \ldots$.*

*Then, $g_0 * g_1 * \cdots * g_{l-1} * (C_l, A_l)$ is a realization of $k^f$ (from $k$).*

**Proof.** Clearly:

$$\begin{aligned} & g_0 * g_1 * \cdots * g_{l-1} * (C_l, A_l) \\ & \quad = (C, A) + (V_{0,\alpha_0}, 0) + g_0 * (V_{1,\alpha_1}, 0) + \cdots + g_0 * g_1 * \cdots * g_{l-2} * (V_{l-1,\alpha_{l-1}}, 0) \end{aligned}$$

is a perturbation of $(C, A)$ which realizes $k^f$. $\square$

But it is not a miniversal realization because the third and sequel terms can affect other entries in $(C, A)$ than the ones in the considered miniversal deformation $\{(C + V_\alpha, A)\}_\alpha$. We illustrate this in the next example.

**Example 3.8.** If we want to find a realization of (4, 4, 4, 4) from (9, 5, 1, 1) using the techniques above, taking one parameter $\alpha$ to go from (9, 5, 1, 1) to (6, 5, 4, 1), $\beta$ to go from (6, 5, 4, 1) to (5, 4, 4, 3) and $\gamma$ to go from (5, 4, 4, 3) to (4, 4, 4, 4) we obtain this result:

$$A_f = \left(\begin{array}{ccccccccc|ccccc|c|c}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&\alpha^{-1}&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\gamma&0&0&0&0&0&0&\alpha^{-1}&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-\alpha&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right).$$

$$C_f = \left(\begin{array}{cccccccc|ccccc|c|c} 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\alpha+\gamma&0&0&0&0&0&\alpha&0&0 \\ 0&0&0&\beta&0&0&0&0&0&0&0&\beta\gamma&0&0&0&1 \end{array}\right).$$

## 4. Minimal realizations

We have seen that the realizations in Example 3.8 are not "minimal". For example, the perturbation parameters can appear in both matrices, whereas the considered miniversal perturbations do not have parameters in the square matrix and place the parameters in predetermined entries in the rectangular one. Our goal is to move the above parameters to the desired entries.

In order to modify the intermediate perturbations, the first step is to study how the parameters in miniversal positions go in the next step to any good sequence.

We undo the change of basis to express a miniversal perturbation $(C_1 + V_\beta, A_1)$, as in Theorem 2.4, of $(C_1, A_1)$ in Proposition 3.1 in the original basis where $(C_1, A_1)$ is expressed as $(C + V_\alpha, A)$.

**Remark 4.1.** Notice that $(V_\beta)_1 = (V_\beta)_{k'_1} = (V_\beta)_{k'_1+1} = (V_\beta)_{l_2} = (V_\beta)_{l_2+1} = \cdots = 0$, since in each block $V_{\beta,i,j}$ there are $max(k'_j - k'_i - 1, 0)$ parameters beginning in the second to the last column.

**Remark 4.2.** From now on we will consider only good sequences. Notice that the second property in Definition 3.6 allows us to take $V_\beta^1 = 0$.

**Proposition 4.3.** *Let $(C_1 + V_\beta, A_1)$ be a miniversal perturbation of $(C_1, A_1)$ with $V_\beta^1 = 0$.*

*If $\bar{V} \doteq (V_\beta^2, 0, V_\beta^3, \ldots, V_\beta^m)^t(e_1, \ldots, e_{k'_1}, e_{l_2-p+1}, \ldots, e_{l_2}, \alpha^{-1}(e_{k'_1-k_2+1} - e_{k'_1+1}), \ldots, \alpha^{-1}(e_{k'_1} - e_{l_2-p}), e_{l_2+1}, \ldots, e_n)$, then:*

1. *$(\bar{V})_{l_j} = 0$, for $j \neq 2$.*
2. *$(\bar{V})_{l_2} = -\alpha^{-1} \sum_{i>2, k_i<k_2+p-1} \beta_{i,2,p} e_i$.*
3. *$(D_1, A) \doteq g * (C_1 + V_\beta, A_1) = (C + V_\alpha + \bar{V}, A)$.*
4. *If $V_\beta^i = 0$ for any $i \neq 2$, then $D_1^i = E_{l_i}$.*

**Proof.** The first two items are a simple consequence of the definition of $\bar{V}$ and Remark 4.1.

Using Proposition 3.1, $g * (C_1 + V_\beta, A_1) = (C + V_0, A) + (G_3 V_\beta G_1^{-1}, G_2 V_\beta G_1^{-1})$ and, from Remark 4.2, $G_2 V_\beta G_1^{-1} = e_{k_1} V_\beta 1 G_1^{-1} = 0$.

On the other hand, from Proposition 3.1, $G_3 V_\beta = (V_\beta^2, \alpha V_\beta^1, V_\beta^3, \ldots, V_\beta^m)^t$ and, using Remarks 3.3 and 4.2, the third item is proved.

Finally, the last item is immediate because $\bar{V}^i = 0$ if $V_\beta^i = 0$. $\square$

The second step is the elimination of the parameters that appear out of miniversal positions.

Firstly we eliminate the parameters in the $l_2$-column.

**Proposition 4.4.** *If $h_1 \doteq diag(I_n, I_m + \alpha^{-1} \sum_{j>2, k_j<k_2+p-1} \beta_{j,2,1} e_j E_2)$, then, $(D_2, A) \doteq h_1 * (D_1, A)$ satisfies that $(D_2)_{l_j} = e_j$ for $j = 1, 2, \ldots, m$.*

**Proof.** It is an easy computation to check that $(D_2)_{l_j} = e_j$ applying the definition of $h_1$ and Lemma 4.3. $\square$

In order to eliminate the remaining parameters out of miniversal positions we introduce some lemmas and definitions which will be useful.

**Lemma 4.5.** *Let $F_q = (f_{q,i,j})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant k} \in \mathbb{C}^{m \times k}$, $q < k$, $(F_q)_k = e_t$, $f_{q,t,q} = f_q$ and $f_{q,t,j} = 0$ for $j > q$.*

*If $T_q \doteq I_k + \sum_{1 \leqslant i < \frac{k}{k-q}} \begin{pmatrix} 0 & 0 \\ (-f_q)^i I_{k-i(k-q)} & 0 \end{pmatrix}$, then:*

1. $T_q^{-1} = I_k + \begin{pmatrix} 0 & 0 \\ f_q I_q & 0 \end{pmatrix}$.
2. $F_{q-1} \doteq F_q T_q$ *has* $(F_{q-1})_j = (F_q)_j$ *if* $j > q$ *and* $f_{q-1,t,q} = 0$.

**Proof.** We prove the first item just multiplying and bearing in mind that

$$\begin{pmatrix} 0 & 0 \\ (-f_q)^i I_{k-i(k-q)} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ f_q I_q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -(-f_q)^{i+1} I_{q-i(k-q)} & 0 \end{pmatrix}$$

if $q > i(k-q)$ and 0 in other case.

The second item is straightforward using that $(T_q)_j = e_j$ for $j > q$ and $(T_q)_q = e_q - f_q e_k$. $\square$

**Lemma 4.6.** *Let $(D, A) \in \mathbb{C}^{m \times n} \times \mathbb{C}^{n \times n}$, where $A$ is a nilpotent matrix with blocks $(k_1, k_2, \ldots, k_m)$ and $(D)_{l_j} = e_j$ for $1 \leqslant j \leqslant m$.*

*Taking the submatrix of $D$ formed by its $l_{s-1}+1, \ldots, l_s$ columns as $F_{k_s-1} \in \mathbb{C}^{m \times k_s}$, if we define $r_s \doteq diag(I_{l_{s-1}}, \prod_{1 \leqslant j < k_s} T_j^{-1}, I_{n+m-l_s})$, then $(D', A) \doteq r_s * (D, A)$ has:*

1. *The block $D'_{s,s} = E_{k_s}$.*
2. $(D')_j = (D)_j$ *if* $j \notin ]l_{s-1}, l_s]$.

**Proof.** The conclusion is obtained by applying Lemma 4.5 and taking into account that $diag(I_{l_{s-1}}, \prod_{1 \leqslant j < k_s} T_j^{-1}, I_{n-l_s}) \cdot A \cdot diag(I_{l_{s-1}}, \prod_{k_s > j \geqslant 1} T_j, I_{n-l_s}) = A$. $\square$

**Definition 4.7.** The partition $(K_1, K_2, \ldots, K_m)$ will be the initial tuple $(k_1, k_2, \ldots, k_m)$ in non-increasing order. We denote $K_{q(i)} := k_i$ and $L_i := \Sigma_{j=1}^{i} K_j$.

**Definition 4.8.** Let $D_{i,j} \in \mathbb{C}^{1 \times k_j}$, we define the Toeplitz square matrix $T(D_{i,j}) \in \mathbb{C}^{k_i \times k_j}$ such that its last row has the $min(k_i, k_j)$ first terms of $D_{i,j}$ (notice that they are the parameters in non-versal positions) and zeros otherwise.

**Lemma 4.9.** *Let $(D, A) \in \mathbb{C}^{m \times n} \times \mathbb{C}^{n \times n}$, where $A$ is a nilpotent matrix with blocks $K_1 \geqslant K_2 \geqslant \cdots \geqslant K_m$, $(D)_{l_j} = e_j$, $D^j$ with zeros in non-versal positions for $1 \leqslant j < s$ and $D_{s,s} = E_{K_s}$.*

*If we define* $\bar{R}_s \doteq \left(\begin{array}{c|c|c} I_{l_{s-1}} & 0 & 0 \\ \hline T(D_{s,1}) \ldots T(D_{s,s-1}) & I_{K_s} & T(D_{s,s+1}) \ldots T(D_{s,m}) \\ \hline 0 & 0 & I_{n-l_s} \end{array}\right)$ *and* $\bar{r}_s \doteq diag(\bar{R}_s, I_m)$, *then,* $(D', A) \doteq \bar{r}_s * (D, A)$ *has:*

1. $D'^j = D^j$ *for* $1 \leqslant j < s$.
2. $D'^s$ *has zeros in non-versal positions.*
3. $(D')_{l_j} = (D)_{l_j}$ *for* $1 \leqslant j \leqslant m$.
4. *If* $D^i = E_{L_i}$, *then* $D'^i = E_{L_i}$.

**Proof.** First of all, $\bar{R}_s A \bar{R}_s^{-1} = A$ since the diagonal blocks of $\bar{R}_s$ are identity matrices and the others are Toeplitz lower triangular matrices.

We obtain $\bar{R}_s^{-1}$ changing the sign of the parameters of $\bar{R}_s$ out of the diagonal. Then, the first item $D'^j = D^j$ for $1 \leqslant j < s$ is straightforward.

Dividing $D$ and $D'$ into blocks $B_{i,j}$ and $B'_{i,j}$ compatible with the identities in $\bar{R}_s$ we have that $B_{2,1} = (D_{s,1} \dots D_{s,s-1})$, $B_{2,2} = E_{K_s}$ and $B_{2,3} = (D_{s,s+1} \dots D_{s,m})$. Then, using that $D' = D\bar{R}_s^{-1}$, $B'_{2,1} = (D_{s,1} \dots D_{s,s-1}) - E_{K_s}(T(D_{s,1}) \dots T(D_{s,s-1}))$, $B'_{2,2} = E_{K_s}$ and $B'_{2,3} = (D_{s,s+1} \dots D_{s,m}) - E_{K_s}(T(D_{s,s+1}) \dots T(D_{s,m}))$ that have zeros in non-versal positions.

By construction it is obvious that $(\bar{R}_s^{-1})_{l_j} = e_{l_j}$ and the third item is proved.

Finally, if $i = s$, we have that $r_s = I_{n+m}$ and if $i \neq s$, $E_{L_i}\bar{R}_s^{-1} = E_{L_i}$ and the last item is proved. □

**Proposition 4.10.** *Let $(D, A) \in \mathbb{C}^{m\times n} \times \mathbb{C}^{n\times n}$, where $A$ is a nilpotent matrix with blocks $K_1 \geqslant K_2 \geqslant \dots \geqslant K_m$ with $(D)_{l_j} = e_j$ and $h_2 \doteq \bar{r}_m * r_m * \dots * \bar{r}_1 * r_1$.*

*Then $h_2 * (D, A)$ has zeros in all non-versal positions.*

**Proof.** It is straightforward. □

Finally, we apply these results to our problem.

**Theorem 4.11.** *Let $(C, A)$ be a pair in BK-canonical form and let $k = (k_1, k_2, \dots, k_m)$ be the tuple of its observability indices with $k_1 > k_2, k_3 \geqslant \dots \geqslant k_m$. Let $(C + V_\alpha, A)$ be the elementary perturbation obtained from the miniversal deformation in Theorem 2.4 taking as unique parameter $\alpha \doteq \alpha_{2,1,p}$, $1 \leqslant p < k_1 - k_2$.*

*Let $(C_1, A_1)$ be a pair in BK-canonical form with $k' = (k_1 - p, k_2 + p, k_3, \dots, k_m)$ as the tuple of its observability indices and let $(C_1 + V_\beta, A_1)$ be a miniversal perturbation of $(C_1, A_1)$ with $V_\beta^1 = 0$.*

*Let $\tilde{q}$ be the permutation that puts the blocks corresponding to the indices $k_1, k_2, \dots, k_m$ in non-increasing order $K_1, K_2, \dots, K_m$.*

*If $\bar{g} \doteq h_2 * \tilde{q} * h_1 * g$ with $g$, $h_1$ and $h_2$ defined as in Propositions 3.1, 4.4 and 4.10, respectively, then:*

1. *$\bar{g} * (C_1 + V_\beta, A_1)$ does not have parameters out of miniversal positions in $(C, A)$.*
2. *If $V_\beta^i = 0$ for any $i \neq 2$, the $K_{q(i)}$-row of the non-square matrix of the pair $\bar{g} * (C_1 + V_\beta, A_1)$ is $E_{L_{q(i)}}$.*

**Proof.** It is an immediate consequence of Lemmas 4.3, 4.6, 4.9 and Proposition 4.10. □

**Corollary 4.12.** *Given a good sequence between partitions $k$ and $k^f$ and their respective BK-canonical forms $(C, A)$, $(C_1, A_1)$, ..., $(C_l, A_l)$, the composition of respective $\bar{g}_i$ as in Theorem 4.11 allows us to obtain a perturbation of $(C, A)$ without parameters out of miniversal positions and in the i-rows for $k_i \in k_{(c)}$.*

**Proof.** The last item in Theorem 4.11 assures that the restriction of considering no parameters in the $i$-rows for $k_i \in k_{(c)}$ is possible because in every step no parameters appear in the rows without entries. □

We will see in the next example how the miniversal realization is better than the one obtained in Example 3.8.

**Example 4.13.** A miniversal realization of (4, 4, 4, 4) from (9, 5, 1, 1) that modifies the realization obtained in Example 3.8 is $(C + V_f, A)$, where $(C, A)$ is an observable pair with indices (9, 5, 1, 1) in BK-canonical form and

$$V_f = \left(\begin{array}{ccccccccc|ccccc|c|c} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\alpha&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&\beta&0&0&0&0&0&0&0&\beta\gamma&0&0&0&0 \end{array}\right).$$

In the above example the miniversal realization obtained in three steps has only three independent entries. This is not true in general as we can observe in the next example.

**Example 4.14.** If we want to find a deformation that goes from (9, 6, 4, 1) to (7, 7, 3, 3), using the techniques of this paper and taking one parameter $\alpha$ to go from (9, 6, 4, 1) to (8, 7, 4, 1), $\beta$ to go from (8, 7, 4, 1) to (7, 7, 4, 2) and $\gamma$ to go from (7, 7, 4, 2) to (7, 7, 3, 3) we obtain that the parameters of a possible miniversal deformation are:

$$V_f = \left(\begin{array}{ccccccccc|cccccc|cccc|c} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\alpha&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\beta&0&0&0&0&0&0&\frac{\beta}{\alpha}&0&0&\gamma&0&0&0 \end{array}\right).$$

That is to say, $(C+V_f, A)$ is a realization of (7, 7, 3, 3) from (9, 6, 4, 1), where $(C, A)$ is an observable pair with indices (9, 6, 4, 1) in BK-canonical form.

If instead of $\frac{\beta}{\alpha}$ we take a different value in the same place, the indices would be (7, 6, 4, 3).

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